Section 4.4 | Finding particular solutions by undetermined coefficients

To solve a non-homogeneous higher order DE
1. Solve the homogeneous case
2. Note the solutions, as the solutions in the particular solution must be linearly independent to the solutions from the homogeneous case
3. If a repeating type pattern occurs with the derivatives of the $f(x)$, then undetermined coefficients method is an option. If not, try variation of parameters. (section 4.6)

Example 1 | Solve $y'' + 3y' + 4y = 3x + 2$

y_c $\quad m^2 + 3m + 4 = 0$

$$m = \frac{-3 \pm \sqrt{9 - 4(1)(4)}}{2(1)} = \frac{-3 \pm \sqrt{-7}}{2} = \frac{-3}{2} \pm \frac{\sqrt{7}}{2} i$$

$$\boxed{y_c = e^{-\frac{3}{2}x}\left[c_1 \cos \frac{\sqrt{7}}{2}x + c_2 \sin \frac{\sqrt{7}}{2}x\right]}$$

y_p $\quad$ pick $y_p = Ax + B$ $\quad \boxed{y_p = \frac{3}{4}x - \frac{1}{16}}$

$y = Ax + B$

$y' = A$

$y'' = 0$

$0 + 3A + 4(Ax + B) = 3x + 2$

$3A + 4Ax + 4B = 3x + 2$

so $4A = 3$ $\quad \boxed{A = \frac{3}{4}}$

$3A + 4B = 2$

$3\left(\frac{3}{4}\right) + 4B = 2$

$4B = 2 - \frac{9}{4}$ $\quad \boxed{B = \frac{-1}{16}}$

$\boxed{\text{Example 2}}$ $\boxed{3y'' + y' - 2y = 2\cos x}$

$\boxed{y_c}$

$3m^2 + m - 2 = 0$

$(3m-2)(m+1) = 0$

$m = \frac{2}{3} \quad m = -1$

$$\boxed{y_c = c_1 e^{\frac{2}{3}x} + c_2 e^{-x}}$$

$\boxed{y_p}$ pick $y_p = A\cos x + B\sin x$

$y_p' = -A\sin x + B\cos x$

$y_p'' = -A\cos x - B\sin x$

$3(-A\cos x - B\sin x) + (-A\sin x + B\cos x)$
$\quad - 2(A\cos x + B\sin x) = 2\cos x$

$-3A\cos x - 3B\sin x - A\sin x + B\cos x - 2A\cos x - 2B\sin x$
$\quad = 2\cos x$

$\cos x(-3A + B - 2A) + \sin x(-3B - A - 2B) = 2\cos x$

$\cos x(-5A + B) + \sin x(-5B - A) = 2\cos x$

$-5A + B = 2$

$-5B - A = 0$

$$\boxed{y_p = \frac{-5}{13}\cos x + \frac{1}{13}\sin x}$$

$-A = 5B$

$A = -5B$

$-5(-5B) + B = 2$

$+25B + B = 2$

$26B = 2$

$\boxed{B = \frac{1}{13}}$

$A = -5\left(\frac{1}{13}\right) = \frac{-5}{13}$

$\boxed{A = \frac{-5}{13}}$

$\boxed{\text{Find the general form of the } y_p \text{ (do not solve)}}$

$\boxed{\text{Example 3}}$ $\boxed{y''' + 9y' = x\sin x + x^2 e^{2x}}$

$\boxed{y_c}$

$m^3 + 9m = 0$

$m(m^2+9) = 0$

$m = 0 \quad m = \pm 3i$

$y_c = c_1 e^0 + e^0(c_2 \cos 3x + c_3 \sin 3x)$

$$\boxed{y_c = c_1 + c_2 \cos 3x + c_3 \sin 3x}$$

$\boxed{y_p}$ $\boxed{\text{pick } y_p = (Ax+B)\sin x + (Cx+D)\cos x + (Ex^2+Fx+G)(e^{2x})}$

$\boxed{\text{Example 4}}$ $\boxed{\text{Find the form of } y_p \quad y'' + 6y' + 13y = e^{-3x}\cos 2x}$

$\boxed{y_c}$ $m^2 + 6m + 13 = 0$

$m = \dfrac{-6 \pm \sqrt{36 - 4(1)(13)}}{2} = -3 \pm 2i$

$$\boxed{y_c = e^{-3x}[c_1 \cos 2x + c_2 \sin 2x]}$$

$\boxed{y_p}$ $y_p = \underbrace{e^{-3x}[A\cos 2x}_{\text{matches part of } y_c \text{ (no LI)}} + B\sin 2x]$

so $\boxed{y_p = xe^{-3x}[A\cos 2x + B\sin 2x]}$

Example 5

If $y_c = 5x^2 + 3x - 1 + e^{2x}$ and
$f(x) = 4x + 5e^{2x} - \cos x$

Set up y_p

first choice $y_p = Ax + B + Ce^{2x} + D\cos x + E\sin x$

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$$\boxed{y_p = x^3(Ax+B) + x(Ce^{2x}) + D\cos x + E\sin x}$$

Section 4.4 Undetermined Coefficients
Extra Practice Problems

Solve each of the following DEs.
Find a particular solution using the method of undetermined coefficients

1. $y'' + 6y' + 8y = -3e^{-t}$
2. $y'' + 3y' - 18y = 18e^{2t}$
3. $y'' + 4y = \cos 2t$
4. $y'' + 7y' + 6y = 3\sin 2t$
5. $y'' + 6y' + 8y = 2t - 3$
6. $y'' + 3y' + 4y = t^3$
7. $y'' - 4y' - 5y = 4e^{-2t}$ $\quad y(0) = 0 \quad y'(0) = -1$
8. $y'' - 2y' + 5y = 3\cos t$ $\quad y(0) = 0 \quad y'(0) = -2$
9. $y'' - 2y' + y = t^3$ $\quad y(0) = 1 \quad y'(0) = 0$
10. $y'' - y' - 2y = 2e^{-t}$
11. $y'' + 9y = \sin 3t$
12. $y'' + 6y' + 9y = 5e^{-3t}$
13. $y'' + 2y' + 2y = 2 + \cos 2t$

(14) $y'' + 25y = 2 + 3t + \cos 5t$

(15) $y'' + 4y' + 3y = \cos 2t + 3\sin 2t$

(16) $y'' + 4y' + 4y = e^{-2t} + \sin 2t$

(17) $y'' + 2y' + y = t^2 e^{-2t}$

(18) $y'' + 3y' + 2y = t^2 e^{-2t}$

$\boxed{\text{Answers}}$

(1) $y = c_1 e^{-2t} + c_2 e^{-4t} + \boxed{2e^{-3t}}$

(2) $y = c_1 e^{-6t} + c_2 e^{3t} + \boxed{3e^{-t}}$

(3) $y = c_1 \sin 2t + c_2 \cos 2t \boxed{-\frac{1}{5}\cos 3t}$

(4) $y = c_1 e^{-6t} + c_2 e^{-t} \boxed{-\frac{21}{100}\cos 2t + \frac{3}{100}\sin 2t}$

(5) $y = c_1 e^{-2t} + c_2 e^{-4t} \boxed{+\frac{1}{4}t - \frac{9}{16}}$

(6) $y = e^{-3/2 t}\left[c_1 \cos\sqrt{7}t + c_2 \sin\sqrt{7}t\right] + \boxed{\frac{1}{4}t^3 - \frac{9}{16}t^2 + \frac{15}{32}t - \frac{9}{128}}$

(7) $y = \frac{-1}{14}e^{-5t} - \frac{1}{2}e^{-t} + \frac{4}{7}e^{-2t}$

(8) $y = e^{t}\left[\frac{-3}{5}\cos 2t - \frac{11}{20}\sin 2t\right] + \frac{3}{5}\cos t - \frac{3}{10}\sin t$

(9) $y = (-23 + 5t)e^{t} + t^3 + 6t^2 + 18t + 24$

(10) $y = c_1 e^{2t} + c_2 e^{-t} + \frac{-2}{3} t e^{-t}$

(11) $y = c_1 \cos 3t + c_2 \sin 3t - \frac{1}{6} t \cos 3t$

(12) $y = c_1 e^{-3t} + c_2 t e^{-3t} + \frac{5}{2} t^2 e^{-3t}$

(13) $y = e^{-t}\left[c_1 \cos t + c_2 \sin t\right] + 1 - \frac{1}{10}\cos 2t + \frac{1}{5}\sin 2t$

(14) $y = c_1 \cos 5t + c_2 \sin 5t + \frac{2}{25} t + \frac{3}{25} + \frac{1}{10} t \sin 5t$

(15) $y = c_1 e^{-3t} + c_2 e^{-t} - \frac{5}{13}\cos 2t + \frac{1}{13}\sin 2t$

(16) $y = c_1 e^{-2t} + c_2 t e^{-2t} + \frac{1}{2} t^2 e^{-2t} - \frac{1}{8}\cos 2t$

(17) $y = c_1 e^{-t} + c_2 t e^{-t} + (t^2 + 4t + 6) e^{-2t}$

(18) $y = c_1 e^{-2t} + c_2 e^{-t} + t\left[-\frac{1}{3} t^2 - t - 2\right] e^{-2t}$